

# A tree-free group that is not orderable

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## Abstract

I. M. Chiswell has asked whether every group that admits a free isometric action (without inversions) on a  $\Lambda$ -tree is orderable. We give an example of a multiple HNN extension  $\Gamma$  which acts freely on a  $\mathbb{Z}^2$ -tree but which has non-trivial generalised torsion elements. The existence of such elements implies that  $\Gamma$  is not orderable.<sup>1</sup>

Let  $\Lambda$  be an ordered abelian group. A group is  $\Lambda$ -free if it admits a free isometric action without inversions on a  $\Lambda$ -tree, and *tree-free* if it is  $\Lambda'$ -free for some  $\Lambda'$ . We refer to the book [2] for a detailed account of the fundamentals of  $\Lambda$ -trees.

In this book Chiswell asks [2, §5.5 Question 3] whether all tree-free groups are orderable, or at least right-orderable. There has recently been some progress made on questions of orderability in tree-free groups. Chiswell himself has shown [4, Theorem 3.8] that  $\mathbb{R}^n$ -free groups are right-orderable. Kharlampovich, Myasnikov and Serbin have shown [7, Corollary 4] that finitely presented tree-free groups are  $\mathbb{R}^n$ -free for some  $n$ ; thus these groups are right-orderable. Chiswell has shown moreover [3, Theorem 4.5] that tree-free groups admit a locally invariant order.

In their recent survey Kharlampovich, Myasnikov and Serbin state [8, Corollary 19] that finitely presented tree-free groups have a finite index subgroup that embeds in a right-angled Artin group: this is a consequence of their result [7, Theorem 2] and the extensive work of Wise (see [11, §16] and [12]) on quasi-convex hierarchies on groups. Since right-angled Artin groups are residually torsion-free nilpotent (see [5, Chapter 3, Theorem 1.1]), it follows that finitely presented tree-free groups are virtually residually torsion-free nilpotent; hence they are *virtually* orderable.

The author has recently [9] raised the question of whether  $\mathbb{Z}^n$ -free groups are residually torsion-free nilpotent. An affirmative answer to this question would have implied that  $\mathbb{Z}^n$ -free groups are orderable.

Nevertheless the answer to Chiswell's question is negative, even when restricted to finitely presented  $\mathbb{Z}^2$ -free groups, as will show presently. It follows that the word 'virtually' cannot be dropped in the discussion above. This suggests an analogy with the situation of braid groups  $B_n$  and their finite index subgroups, the pure braid groups  $P_n$ : the former are right-orderable but not orderable (see [10, §4]), while the latter are residually torsion-free nilpotent [6].

Recall that a group  $G$  is *orderable* if there is a linear order  $\leq$  on  $G$  satisfying  $x \leq y \Rightarrow gxh \leq gyh$  for  $g, h \in G$ . (One can define right-orderable by restricting to  $g = 1$  in the definition above.) It is well-known and easy to see that in an orderable group  $G$  there can be no non-trivial *generalised torsion elements*: these are elements  $g$  such that  $g^{h_1}g^{h_2}\dots g^{h_n} = 1$  for some  $h_1, \dots, h_n \in G$  and  $n \geq 1$ . (Here  $g^h$  denotes the conjugate  $h^{-1}gh$ .)

Let  $F$  be the free group on  $\{x, y, z\}$ , and consider the natural free action of  $F$  on the corresponding Cayley graph, viewed as a  $\mathbb{Z}$ -tree. Observe that  $xy^{-1}$ ,  $yz^{-1}$  and  $zx^{-1}$  and their respective inverses belong to distinct conjugacy classes since they are cyclically reduced as elements of  $F$  and none is a cyclic permutation of another. Moreover, the translation lengths of these elements are all equal to 2.

Now taking  $s_1 = xy^{-1} = s_2$ ,  $t_1 = yz^{-1}$ ,  $t_2 = zx^{-1}$ ,  $u = u_1$  and  $v = u_2$ , and applying [1, Proposition 4.19], the multiple HNN extension

$$\begin{aligned}\Gamma &= \langle x, y, z, u_1, u_2 \mid u_1 s_1 u_1^{-1} = t_1, u_2 s_2 u_2^{-1} = t_2 \rangle \\ &= \langle u, v, x, y, z \mid u(xy^{-1})u^{-1} = yz^{-1}, v(xy^{-1})v^{-1} = zx^{-1} \rangle\end{aligned}$$

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is seen to be  $\mathbb{Z}^2$ -free. However,

$$1 = (xy^{-1})(yz^{-1})(zx^{-1}) = xy^{-1} \cdot u(xy^{-1})u^{-1} \cdot v(xy^{-1})v^{-1},$$

whence  $xy^{-1}$  is a non-trivial generalised torsion element of  $\Gamma$ , and  $\Gamma$  is not orderable. This gives the promised negative answer to Chiswell's question.

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